**Ornstein-Uhlenbeck Process**

Now gonna work through a whole bunch of examples,

**Example 1**

First the easiest:



We can integrate it to obtain,



Let’s take X(t) = at + bW(t). Then we see that:



And so now we could more or less straightforwardly ascertain the probability distribution of X. For instance,



which is consistent with our other calculations. Can we get a FP equation, w/o using that formalism above? So I guess there are two different ways to write P(ξ,t+dt).



I think we need to use the former in this case? Proceeding,



And so we have:



How would we solve that?

**Example**

A liquor store thief stumbles out of the store and travels in a straight line. His probability of taking a 1m step forward is 0.7, and is probability of taking a 0.5m step backwards is 0.3. He takes a step every second. Model this as an SDE. So,



dXi is binomially distributed. The average and variance of dXi is:



So seems that we can model ΔXi as:



where ΔWi is a Gaussian unit normal variable. If we go to the continuum limit, then, we should have:



Thus, after 1s, the average displacement would be 0.55t = 0.55, and the variance would be 0.473t = 0.473.

**Example 2**

Now one that commonly shows up in the physics of Brownian motion.



(where X would be velocity) We can formally solve this equation for X(t), assuming some specific input w(t), using GF’s. And note the Ito and Stratonovich versions of this SDE are identical. I think the result is:



Can we evaluate the integral? Well, yes. We have a constant plus a Gaussian variable. Recalling,



we can say,



We should be able to combine these two to say,



Could also write as:



i.e., a normally distributed variable with given mean and variance. Just to be sure, let’s calculate directly the mean and variance.



And,



So that,



and,



So that agrees! To be even surer, let’s consider the differential:



So it does satisfy the ODE. What would a Taylor series expansion of X look like? Well it wouldn’t be in powers of dt alone, but also dW. Can see that to first order in t, we’d have (taking W(0) = 0):



I suppose that even simply follows from the ODE. Let’s look at the probability distribution function. It ought to be:



That’s interesting. Now let’s do the FP equation:



Looks pretty gross. Are there PDE methods to solve this? BTW, another interesting way to get the variance is to work out an equation for X2. Recalling,



Can do,



Take expectation of both sides:



The <dW> goes away because it’s independent of X in the Ito formulation. So we can solve this equation:



At t = 0, this is X02, so



Then we need <X>2. By taking the expectation of the SDE, we see:



Solution is:



And finally,



So yeah!

**Example 3**

Let’s move to another differential equation:



Let’s do a Taylor series expansion,



Keeping just to order t,



Now let’s consider taking the small t limit. In this limit it would seem that W2 → Ddt, in which case we’d return to the original SDE.

Now to solve the ODE, we would think we could divide by X, but as we know, we don’t necessarily have d(lnX)/dt ≠ X-1dX/dt. So the procedure we used earlier, treating w as a real variable, doesn’t always work. It only worked because the Ito SDE happened to be the same as its Stratonovich version. So let’s consider. If we assume that X = F(W,t), then it follows from the general rule dF/dt = ∂F/∂t + ∂F/∂W∙w + (D/2)∂2F/∂W2 that a(X,t) = ∂X/∂t + (D/2)∂2X/∂W2, and b(X,t) = ∂X/∂W. So then we would propose:



Which leaves us with the same result. More specifically, say we had supposed X = eG(W,t).



Now even the ∂G/∂t term could have a W in it. So there’s no point in equating coefficients. But let’s suppose ∂G/∂W = b nonetheless. Then G = bW + ψ(t). Filling this in, we’d have:



So our solution is:



Imposing initial conditions, we’d arrive at:



Another possibility here is to translate it to the Stratonovich version, for which ‘ordinary’ calculus applies. So, doing this by hand again for practice,



(and this matches our general formula at the top of the file) Then we can do standard manipulations,



which is the same as before. Note the Taylor expansion is:



and this matches. In retrospect, I can see now how treating the preceeding linear SDE as an ODE worked out as its Stratonovich version was the same as the Ito version. Now let’s look at the probability distribution:



Interesting. This is a ln-normal distribution.

**Example 3.6**

Let’s consider the GDMPK SDE:



Let’s change variables to λ = sinh2x. But before we do, we have to convert to Stratonovich version using,



So,



Now proceeding with the change of variables,



Filling these in,



Converting back to Ito,



we have the same thing, since the W term is independent of x. :



**Example 3.7**

Let’s consider the GDMPK SDE:



Let’s change variables to λ = sinh2x. But before we do, we have to convert to Stratonovich version:



Now proceeding with the change of variables,



Filling these in,



Since γij = 2Kij/Kii, can say,



Since the dW term is independent of x, converting back to Ito gives us the same thing.

**Example 4**

Let’s do a matrix example. So our equations are:



To solve, we need to put in Stratonovich form,



and,



So we have:



So for short:



One option is to go to the interaction picture,



Then (and note this product rule would not work unless we were in Stratonovich form)



and, then,



We can write this in matrix form as, trying to remind ourselves that w(t) is a time-dependent variable:



Does **B** commute with itself?



So we can write:



So then,



Now to evaluate this matrix, we need to diagonalize it?



So,



And so,



and so….



So,



**Example 5**

Consider:



(summation implicit) Let’s use GF again. Assume a solution of the form,



Plugging this into the equation we have:



From which it appears that (intepreting as matrices now)



I think we can eliminate the w, because it’s got whatever d.o.f., and then the B by multiplying both sides by its inverse.



So solution is:



I think this can be done, in principle, just like we did for the 1D case above, where now the wk are independent variables and can be added as independent normal variables.

**Example 6**

Suppose we wanted simply an expression for the evolution of the average of X, where dX/dt = a(X) + b(X)w. Starting from the probability distribution, we’d have:



and this agrees with the ODE since:



Note we can separate the averages in the last term since b(X(t)) depends on w(t) for all times up to but not including t. We have a similar relationship proceeding from the integral expression:



We can neglect ∫b(X)dW because these are independent random variables in a sense. What about the variance?



And from direct differentiation:



Taking the expectation of both sides we have:



And this matches!

**Series solution of SDE’s**

We’re interested in a Taylor series expansion of a stochastic variable since we often cannot solve the differential equation, or for the probability distribution function, discussed below. Let’s start with a deterministic example first. Consider that we can write (note this can’t be read as simply recursively substituting X(t) into D[ ], ‘cause it doesn’t work out; rather I’m just taking D(X(t)) and using the same kind of identity I used on X(t), that it’s equal to its initial value plus the integral from 0 to t):



This is a kind of nice derivation of the Taylor series expansion + remainder term. So then, suppose we have the stochastic equation:



We start by writing wj(t)dt = dWj(t). And then, we make manipulations similar to what was done before. I’ll stop at ‘second’ order.



(noting ∫dW = W) Next we work out these derivatives:



and,



Then we plug these into the expansion. And we’d do the f(x,t) = f(x(0),0) + ∫df/dt∙dt thing again. Suppose we got the following coupled differential equations. A series expansion of these guys would be a lot easier, since the differential operator is *linear*. We can do it like is done for the propagator in QM.



Integrating:



Filling the result back into the equations:



Simplifying, and keeping only up to order z,



Now suppose, just suppose, we have the following ODE’s (I’m going to make the dots implicit):



Let’s integrate



and then we would need to develop an expansion for the integrands. These are just the u’s themselves, and we already have the expansion right in front of us. So we plug these back into the integrals.



keeping only terms up to order z, keeping in mind that u+ ~ 1, and u- ~ √z



Filling in u+(0) = 1, then we have:



Working this out,



Doing same for u-, we have after filling the u’s back in:



keeping only up to ~ z, and filling in u+(z) ~ 1 as appropriate:



Perhaps we could afford to be less meticulous if we used the Stratonovich version of the differential?